Monte-Carlo simulation of particle acceleration in braided magnetic fields

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Abstract

Supernova remnants are expected to contain braided (or stochastic) magnetic fields, which are in some regions directed mainly perpendicular to the shock normal. For particle acceleration due to repeated shock crossings, the transport in the direction of the shock normal is crucial. The mean squared deviation along the shock normal is then proportional to the square root of the time. This kind of anomalous transport is called *sub*-diffusion. We use a Monte-Carlo method to examine this non-Markovian transport and the acceleration. As a result of this simulation we are able to examine the propagator, density and pitch-angle distribution of accelerated particles, and especially the spectral properties. These are in broad agreement with analytic predictions for both the *sub*-diffusive and the diffusive regimes, but the steepening of the spectrum predicted when changing from diffusive to *sub*-diffusive transport is found to be even more pronounced than predicted.

1 Introduction

The acceleration of high energy particles in astrophysical plasmas is a transport process in configuration and momentum space. In describing the acceleration of charged particles in a magnetised plasma, most analytical descriptions of this process are based on the assumption that the phase-space density $f(\boldsymbol{x},\boldsymbol{p},t)$ is to zeroth order isotropic and independent of the pitch angle $\mu = \cos\alpha = \boldsymbol{p}\cdot\boldsymbol{B}/(p\,B)$ between the particle momentum $\boldsymbol{p}$ and magnetic field $\boldsymbol{B}$. Under this assumption, the process of acceleration at a plane shock wave moving in x-direction can be described using the isotropic particle density $n(x,p,t) = 4\pi\,p^2\,f^{(0)}(x,p,t)$, where $p = |\boldsymbol{p}|$. The transport equation in a plasma at velocity $u(x,t)$ is then given by (e.g. Parker 1965; Jones & Ellison 1991):

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(u\,n + F) = \frac{1}{3}\frac{\partial u}{\partial x}\frac{\partial}{\partial p}(p\,n)\,, \tag{1}$$

where $F(x,p,t)$ is the flux due to the stochastic propagation of particles in configuration space. If the spatial transport can be described by standard diffusion, then F is proportional to the gradient in the density, and Eq. (1) is the well known diffusion-convection equation. In this case, the momentum dependence of the phase-space density of particles accelerated at a strong shock is given by a power law $f(p) \propto p^{-s}$ with spectral index $s = 3r/(r-1)$, depending solely on the compression ratio[1] r of the shock. However, the presence of a braided magnetic field (Jokipii & Parker 1969), can introduce a non-diffusive spatial transport. This is important especially in quasi-perpendicular shock fronts, where the mean magnetic field $\boldsymbol{B}_0$ lies in the plane of the shock, and a stochastic component with $\delta b := \langle|\delta\boldsymbol{B}|\rangle/|\boldsymbol{B}_0| \ll 1$ parallel to the shock normal exists (in x-direction). Particles which follow the field lines are subject to a combined diffusion process. One is along the field line due to pitch-angle scattering and the other is introduced by the stochastic spatial fluctuations of the magnetic field on a larger scale as those responsible for scattering. This together leads to an anomalous transport of particles while gaining energy due to shock crossings, which is outlined in Sect. 2, followed by a brief description our Monte-Carlo method in Sect. 3. This method is designed to investigate test-particle acceleration in magnetic fields with a stochastic component, without a priory assumptions about the pitch-angle distribution of the phase-space density. The results are presented in Sect. 4, showing especially the dependence of the spectral index s on the compression ratio r in two different transport regimes in comparison to analytical treatments.

[1] $r = \rho'/\rho$ with ρ' and ρ are the downstream and upstream plasma densities respectively.

2 Anomalous transport

The main aspect of particle transport in a braided field (stochastic field with $\delta b \ll 1$) is the introduction of memory to the particle propagation. The change of the density at time $\tilde{t}$ is no longer proportional to the second derivative of the density at this time alone (standard diffusion equation), but also depends on the second derivative at times $t < \tilde{t}$. This arises, because any local variation of the particle density which is caused by the geometry of the magnetic field itself is *not* the source of a diffusive particle flux, and remains associated with the field line. This contribution has therefore to be subtracted from the standard diffusion term. A formulation of this kind of anomalous transport has been given by Balescu (1995). Effectively this introduces a memory of the particle density at time $\tilde{t}$, on the spatial realisation of the magnetic field to which a particle was correlated during $\tilde{t} - t_{\rm corr} < t < \tilde{t}$. An important consequence of transport in braided fields is revealed by the time dependence of the mean quadratic deviation perpendicular to $\boldsymbol{B}_0$, in x-direction $\langle(\Delta x(t))^2\rangle \propto t^\alpha$, which is given by $\alpha = 1/2$ (Rechester & Rosenbluth 1978; Rax & White 1992). This kind of transport is called *sub*-diffusion. Note that here x is the direction along the shock normal, and the relevant reference system is the magnetic field, which flows downstream with the background plasma. The time dependence shows that particles are even more effectively swept away from the shock in downstream direction as compared to standard diffusion ($\alpha = 1$). This increases the escape probability and leads to a steeper spectrum as shown by the results in Sect. 4.

3 Monte-Carlo method

The simulation of particle acceleration in a stochastic field (static in the background plasma) has to consider the memory introduced by the magnetic field as described in the previous section. The spatial transport is a non-Markovian process. We generate a constant mean magnetic field and stochastic fluctuations at equidistant grid points, and assume the field to be linear in between. Using a random number generator for the stochastic fluctuations, which allows to recall all values, we are able to assure a complete memory of the field until the particle crosses an escape boundary far downstream (Gieseler et al. 1997). At the same time this method allows to use a new random number for each field patch the particle crosses, which leads to the standard diffusion. A combination of the recalled value and a new random component would simulate a finite correlation time of particle and magnetic field. This is, of course, the more realistic case. However, to investigate the principal effect of *sub*-diffusion, we present here only results of 'pure' *sub*-diffusion and standard diffusion. Particles move along the field lines under the influence of pitch-angle scattering. The length scale of the grid spaces of the field sampling is chosen to be in the same order as the scattering length. This assures, that while particles are transported in configuration space due to the field line geometry, they diffuse along the field line itself. At the same time this avoids, that particles diffuse along the field while sampling only a linear patch of it. At a change of the magnetic field direction (in particular at the shock) we make use of the conservation of the magnetic moment $p_\perp^2/B$, where $p_\perp$ is the component of the momentum perpendicular to the magnetic field B. This approximation is valid especially for non-relativistic quasi-perpendicular shocks,

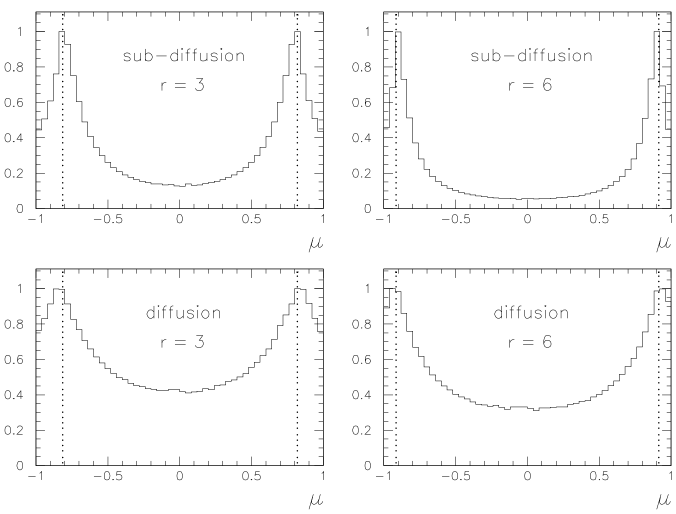

Figure 1: *Upstream pitch-angle distribution for* sub-*diffusion (upper) and diffusion (lower), for two compression ratios. The dotted lines indicate an approximation of the maximal pitch angle $|\mu|$ for which reflection off the shock is possible.*

which we consider here (Gieseler et al. 1999, see also for a description of pitch-angle scattering). The momentum remains constant in the corresponding upstream and downstream rest frames. On crossing the shock the momentum and pitch-angle is transformed into the new system (Gieseler 1998). This method allows to measure the particle propagator and the steady state density profile, which are in agreement with theoretical predictions from Kirk et al. (1996) for *sub*-diffusive transport (Gieseler et al. 1997). Furthermore we are able to measure the pitch-angle distribution and the momentum spectrum which are presented in the next section.

4 Particle acceleration at quasi-perpendicular shocks

In accelerating particles over between two orders of magnitude (for the steepest spectra) and six orders of magnitude we always find a power law for the momentum distributions. We do not include loss mechanisms, and fit a power law function $f \propto p^{-s}$ between about one order of magnitude above the injection momentum and one order of magnitude below the (technical) cut-off. The results are plotted in Fig. 2 for relativistic particles ($v = c$) at non-relativistic shocks ($u_{\rm s} \ll c$) for various compression ratios r. Dots represent standard diffusive acceleration, where the value of the fluctuation of a patch of field line is always random, i.e. no memory effect is introduced. The stars show the spectral index for particles which move always along the same field line, so that *sub*-diffusive behaviour can take effect. The statistical error of the fit itself is well represented by the marker symbols. However, whereas the flatter diffusive spectra extend over many orders of magnitude, the steep *sub*-diffusive spectra are more difficult to measure. The maximal systematical error in finding the spectral index from the momentum distribution is indicated by error bars. Because the memory effect for *sub*-diffusion can not set in immediately, the momentum distribution has a plateau below about ten times the injection momentum. This is indicated by the lower bound of the error bar. A fit to the region where the spectrum is cut off due to technical reasons gives the upper bound of the error bar. For spectra flatter than about $s = 5$, a cut-off is effectively absent, so that the upper bound almost coincides with the plotted index. It can be seen from Fig. 2, that the spectrum for *sub*-diffusive acceleration is significantly steeper than for standard diffusion. We now compare our results to analytical predictions, remembering that these are found under the assumption of an almost isotropic pitch-angle distribution. For standard diffusion the result was referred in connection with Eq. (1): $s = 3r/(r-1)$, and plotted as a dashed line in Fig. 2. Although we found the pitch-angle distribution is not really isotropic in this case, the spectral index found by the Monte-Carlo method agrees quite well with the analytical result. For *sub*-diffusive transport, an analytical solution was found by Kirk et al. (1996):

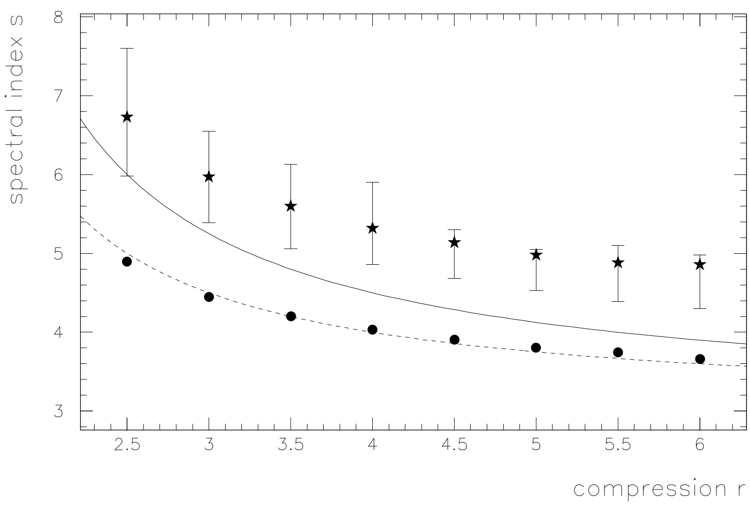

Figure 2: *Spectral index s vs. compression ratio r. Discrete symbols represent our Monte-Carlo results. Lines represent analytical results for isotropic phase-space distributions (see text). Stars and solid line:* sub-*diffusion. Dots and dashed line: diffusion.*

$$s = 3\left(1 + \frac{n(-\infty)}{n(0)}\frac{1}{r-1}\right) = \frac{3r}{r-1}\left(1 + \frac{1}{2r}\right)\;; \quad \text{where} \quad \frac{n(0)}{n(-\infty)} = \frac{2}{3}\,. \tag{2}$$

The second relation means that the density of continuously injected particles at the shock is less than the density far downstream. The resulting spectral index $s(r)$ is plotted as a solid line in Fig. 2. Again, this result was found under the assumption of an almost isotropic phase-space density. The Monte-Carlo method does not make any assumptions on this distribution, moreover we are able to measure the pitch-angle distribution at any distance from the shock. Figure 1 shows the pitch-angle distribution immediately upstream of the shock, in the upstream rest frame for the *sub*-diffusive and diffusive transport regime, at compression ratio $r = 3$ and $r = 6$. Especially for *sub*-diffusive transport and high compression ratio, we found the highest anisotropy. Here, the deviation of the Monte-Carlo results from the analytical result (2) is most prominent (see Fig. 2). We found, that the density of accelerated particles is not only reduced at the shock by the amount predicted by Kirk et al. (1996), moreover a jump arises at the shock, which is intimately related to an anisotropic phase-space distribution (Gieseler et al. 1999). This jump is such, that the upstream density is even more reduced, than indicated by Eq. (2) (Gieseler 1998). This leads to an increased escape probability, and therefore to a steeper spectrum, as compared to the analytical result.

5 Conclusions

We presented Monte-Carlo simulations of particle acceleration at non-relativistic quasi-perpendicular shock fronts. We found that a stochastic component in addition to the mean magnetic field introduces *sub*-diffusive particle transport. The transport aspects (like propagator and density) were compared to analytical treatments earlier (Gieseler et al. 1997), and we found very good agreement. Moreover, we tested our Monte-Carlo code for oblique shocks against semi-analytical results, and found precise agreement again (Gieseler et al. 1999). Here we showed, that particle acceleration under the *sub*-diffusive transport regime leads to a much steeper spectrum (e.g. $s = 5.3$ for $r = 4$) compared to standard diffusion ($s = 4.0$ for $r = 4$), even steeper than predicted by Kirk et al. (1996). The steepening of the spectrum depends strongly on whether or not particles are correlated to field lines, and not (to first order) on the shock velocity, the scattering operator, or the amplitude of the magnetic field fluctuations. However, if the mean field is not strictly perpendicular, i.e. is oblique with an angle Θ with respect to the shock normal, then the transport properties depend on the amplitude of the fluctuations. The *sub*-diffusive transport will take effect as long as $\delta b > 1/\tan\Theta$. It is clear, that for the *sub*-diffusive transport regime our result yields an upper limit on the spectral index (i.e. the steepest possible), because it was produced by an unlimited correlation of particle and field line. In reality, particles will, of course, decorrelate from a given initial field geometry. This is connected with the realisation of the magnetic field itself, and is subject to further investigation.

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